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FREE TORSIONAL VIBRATIONS OF AN ELASTIC CYLINDER WITH LAMINATED PERIODIC STRUCTURE†

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Abstract—The theory of torsional vibrations of a circular cylinder, with a periodic variation of elastic constants and density normal to the axis of the cylinder, is developed in terms of Floquet waves. Floquet waves are quasi-periodic waves, whose amplitude profile has the same periodicity as that of the material and repeats with the periodicity of the cell. Using Floquet's theory, the dispersion spectrum is obtained for time-harmonic waves propagating in a laminated cylinder with periodic structure. It is shown that the dispersion spectrum has a band structure, consisting of passing bands and stopping bands. Motion in the case of grazing incidence, and motion at the end of the zones is discussed. It is also shown that as the radius of the cylinder tends to infinity, the torsional waves in a circular cylinder degenerate to SH-waves in laminated plates.

1. INTRODUCTION

Free torsional motions of homogeneous, isotropic, linearly elastic circular cylinders have attracted attention early in the development of continuum dynamics. Indeed, it appears that a first study of this problem, as reported by Rayleigh[1], was carried out by Chladni[2] almost two centuries ago. The problem is discussed in fairly complete form by Love[3].

The early attention given to the problem is explained by the relative simplicity of the analytical formulation. With reference to a cylindrical system of coordinates, only one component of the displacement vector is non-vanishing and so is only one component of the stress tensor. Further simplification is due to the possibility of separating the radial and the axial dependence.

The relative simplicity of the analytical treatment of the above problem suggests that generalizations to cylinders of non-homogeneous composition might also be tractable. One such generalization is considered in the present study by assuming that the cylinder is fabricated from a periodically-layered two-phase composite, the axis of the cylinder being normal to the planes of the layers. The two materials which make up the two-phase composite have two different elastic properties and two different mass densities, and each layer is in perfect bond with adjacent ones.

The motivation for the present study is at least two-fold. First, there exists a possibility that the frequency spectrum describing free-harmonic vibrations of the composite cylinder might be such as to be potentially useful as a device in microwave technology. Indeed, torsional motions are being used in high frequency filter circuits and delay lines and the system to be studied, which does not appear to have been fully investigated here-to-fore, might offer improvements. Secondly, a number of efforts have been undertaken in recent years to construct a variety of approximate continuum theories to describe the dynamic behavior of composites. The availability of exact solutions of problems based on the theory of elasticity for a bounded body will be most useful in assessing the merits of the various approximate theories which had been proposed.

The governing equations of the problem are set down in Section 2, where the frequency equation for the free motion of a periodically structured cylinder is also derived. Section 3 is devoted to solving the governing frequency equation for Floquet waves, including a discussion of several degenerate cases. The presentation and a discussion of the resulting frequency spectrum is also given in this section. Section 4 deals with the mode shapes at the two end-points of the Brillouin zone. In Section 5 we treat the limiting case of SH-waves in laminated plates.

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2. GOVERNING RELATIONS

Consider an infinite circular cylinder of radius a with material properties which vary periodically along the length of the cylinder, as shown in Fig. 1. We assume that in the region $0 < z < l$, λ and μ are the two Lamé's constants and ρ is the mass density. The corresponding constants in the region $-l' < z < 0$ are assumed to be λ', μ' and ρ', respectively. The region $(r, z: r = a, -l' < z < l)$ defines a typical unit cell consisting of two laminae of two different material properties, and of lengths l and l', so that the lattice distance of the unit cell is $d \equiv (l + l')$. We assume that an infinite number of such unit cells are bonded together at their common interfaces $z = (l + nd)$ and $z = -(l' + nd)$, $n = 0, 1, 2, \ldots, \infty$ so that the region $\{0 < r \leq a, -(l' + nd) \leq z \leq (l + nd)\}$ now defines an infinite circular cylinder with periodic structure of material constants, with period d.

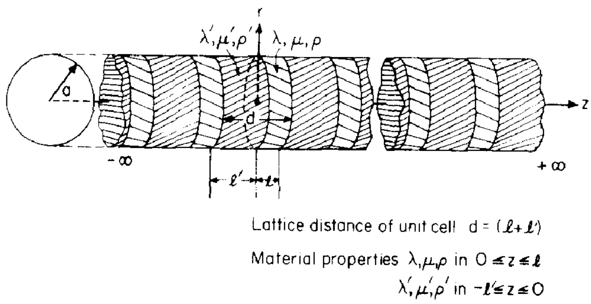

Fig. 1. Infinite circular cylinder with periodic structure of material constants.

Homogeneous cylinder

We consider a homogeneous cylinder first. In orthogonal cylindrical coordinate system (r, θ, z) the three-dimensional equations of motion of the linear theory of elasticity, in the absence of body forces, are

$$(\lambda + 2\mu)\frac{\partial}{\partial r}\Delta - 2\mu\frac{\partial}{r\partial\theta}\,\omega_{r\theta} + 2\mu\frac{\partial}{\partial z}\,\omega_{zr} = \rho\ddot{u}_r,$$

$$(\lambda + 2\mu)\frac{\partial}{r\partial\theta}\Delta - 2\mu\frac{\partial}{\partial z}\,\omega_{\theta z} + 2\mu\frac{\partial}{\partial r}\,\omega_{r\theta} = \rho\ddot{u}_\theta,$$

$$(\lambda + 2\mu)\frac{\partial}{\partial z}\Delta - 2\mu\frac{\partial}{r\partial r}(r\omega_{zr}) + 2\mu\frac{\partial}{r\partial\theta}\,\omega_{\theta z} = \rho\ddot{u}_z, \tag{1}$$

where u_r, u_θ, u_z are the three components of the displacement in the radial, tangential and axial directions, respectively[3]. In addition, Δ is the dilatation and $\omega_{r\theta}$, $\omega_{\theta z}$, ω_{zr} are the three components of the rotation tensor which satisfy the identical relation

$$\frac{\partial}{\partial r}(r\omega_{\theta z}) + \frac{\partial}{\partial\theta}\,\omega_{zr} + r\frac{\partial}{\partial z}\,\omega_{r\theta} \equiv 0.$$

In terms of the displacement components, the three components of the antisymmetric rotation tensor are

$$\omega_{\theta z} = \frac{1}{2}\left(\frac{\partial}{r\partial\theta}\,u_z - \frac{\partial}{\partial z}\,u_\theta\right),$$

$$\omega_{zr} = \frac{1}{2}\left(\frac{\partial}{\partial z}\,u_r - \frac{\partial}{\partial r}\,u_z\right),$$

$$\omega_{r\theta} = \frac{1}{2r}\left[\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial}{\partial\theta}\,u_r\right], \tag{2}$$

and the scalar Δ is given by

$$\Delta \equiv \frac{\partial}{r\partial r}(ru_r) + \frac{\partial}{r\partial\theta}\,u_\theta + \frac{\partial}{\partial z}\,u_z. \tag{3}$$

In this coordinate system, Hooke's law for a homogeneous isotropic medium assumes the form

$$\tau_{rr}=\lambda\Delta+2\mu\frac{\partial}{\partial r}u_r,$$
$$\tau_{r\theta}=\mu\left[\frac{\partial}{r\partial\theta}u_r+r\frac{\partial}{\partial r}\left(\frac{1}{r}u_\theta\right)\right],$$
$$\tau_{rz}=\mu\left(\frac{\partial}{\partial z}u_r+\frac{\partial}{\partial r}u_z\right),$$
$$\tau_{\theta z}=\mu\left(\frac{\partial}{\partial z}u_\theta+\frac{\partial}{r\partial\theta}u_z\right), \tag{4}$$

and the six components of the strain tensor are given by

$$e_{rr}=\frac{\partial}{\partial r}u_r, \qquad e_{r\theta}=\frac{1}{2}\left[\frac{\partial}{r\partial\theta}u_r+r\frac{\partial}{\partial r}\left(\frac{1}{r}u_\theta\right)\right],$$
$$e_{\theta\theta}=\frac{\partial}{r\partial\theta}u_\theta+\frac{1}{r}u_r, \quad e_{\theta z}=\frac{1}{2}\left(\frac{\partial}{\partial z}u_\theta+\frac{1}{r}\frac{\partial}{\partial\theta}u_z\right),$$
$$e_{zz}=\frac{\partial}{\partial z}u_z, \qquad e_{rz}=\frac{1}{2}\left(\frac{\partial}{\partial r}u_z+\frac{\partial}{\partial z}u_r\right). \tag{5}$$

The smooth cylindrical surface of the prismatic cylinder with generators parallel to the z-axis is assumed to be free of surface tractions. Hence the surface conditions are

$$\tau_{rr},\tau_{r\theta},\tau_{rz}=0 \quad \text{on} \quad r=a, -\infty<z<\infty. \tag{6}$$

In this paper our main interest is the study of quasi-periodic torsional motion of the prismatic cylinder of uniform circular cross-section whose material properties vary periodically with length. For motion which is harmonic with respect to time, we assume existence of solutions in the form of torsional motions

$$u_\theta=V(r)\exp i\left(\frac{\pi}{2a}\xi z-\omega t\right),$$
$$0<z<l$$
$$0<t$$
$$u_r=0 \quad \text{and} \quad u_z=0, \tag{7}$$

where ξ is the wave-number in the longitudinal direction, ω is the circular frequency, t is the time and a is the radius of the cylinder.

It therefore follows that in torsional motion, the kinematic and dynamic variables take a simpler form and are explicitly given by

$$\Delta\equiv 0, \qquad \omega_{zr}\equiv 0,$$
$$\tau_{rr}=\tau_{rz}\equiv 0, \qquad \omega_{\theta z}=-\frac{1}{2}\frac{\partial}{\partial z}u_\theta,$$
$$\tau_{r\theta}=\mu r\frac{\partial}{\partial r}\left(\frac{1}{r}u_\theta\right), \quad \omega_{r\theta}=\frac{1}{2r}\frac{\partial}{\partial r}(ru_\theta),$$
$$\tau_{\theta z}=\mu\frac{\partial}{\partial z}u_\theta. \tag{8}$$

Simultaneously the equation of motion $(1)_2$ simplifies to the form

$$2\mu\left(\frac{\partial}{\partial r}\omega_{r\theta}-\frac{\partial}{\partial z}\omega_{\theta z}\right)=\rho\ddot{u}_\theta, \tag{9}$$

and the remaining two equations of the set are satisfied identically. It is now apparent that torsional motion is clearly equivoluminal and eqn (9) is a statement of the law of balance of angular momentum. In terms of displacement component u_θ this equation can be written as

$$\mu\left(\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}ru_\theta+\frac{\partial^2}{\partial z^2}u_\theta\right)=\rho\ddot{u}_\theta. \tag{10}$$

With solutions of the form (7), eqn (10) reduces to a second-order ordinary differential equation

$$\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}rV(r)+\left(\frac{\pi}{2a}\chi\right)^2V(r)=0, \tag{11}$$

where

$$\chi=(\Omega^2-\xi^2)^{1/2}, \tag{12}$$

$$\Omega=\omega/\omega_s,\quad \omega_s=\frac{\pi}{2a}(\mu/\rho)^{1/2}, \tag{13}$$

and Ω is the dimensionless frequency and ξ, χ are the dimensionless wave-numbers in the longitudinal and radial directions, respectively. The reference frequency ω_s is chosen to be the lowest antisymmetric thickness-shear frequency of a homogeneous, isotropic, infinite plate of thickness $2a$.

Equation (11) is a Bessel equation of order one, and for a solid cylinder the *primitive* of the differential equation is

$$\begin{aligned}V(r)&=DJ_1\left(\frac{\pi}{2a}\chi r\right)\quad \text{for}\quad \Omega>\xi,\\&=Dr\qquad\qquad\quad\ \text{for}\quad \Omega=\xi,\end{aligned} \tag{14}$$

where in the first case, the radial eigenvalue χ is real and in the second case $\chi=0$. The second case corresponds to *pure* torsion and the differential eqn (11) has a turning point. It is not necessary to consider this case separately as the turning point solution can be obtained from the first case by using d'Alembert's limiting procedure[4]. A third situation appears to arise when $\Omega<\xi$, i.e. when χ is imaginary. In this case the solution is in the form of modified Bessel functions. However, we will soon see that imaginary values of χ are inadmissible in the present problem.

The cylindrical surface $r=a$ is free of traction and therefore the boundary conditions (6) must be satisfied. With the displacement solution given by (14), the stresses τ_{rr} and τ_{rz} vanish identically and the remaining shear stress $\tau_{r\theta}$ vanishes on the surface $r=a$, only when χ is a root of the equation

$$\frac{\partial}{\partial r}\left[\frac{1}{r}J_1\left(\frac{\pi}{2a}\chi r\right)\right]_{r=a}=0. \tag{15}$$

This leads us to the transcendental equation

$$2J_1\left(\frac{\pi}{2}\chi\right)-\left(\frac{\pi}{2}\chi\right)J_0\left(\frac{\pi}{2}\chi\right)=0, \tag{16}$$

which has a denumerable infinity of real and *simple* zeros χ_n where $\chi_0<\chi_1<\chi_2<\cdots<\chi_n<\cdots$. The first few roots of this transcendental frequency equation are[5]

$$\chi_0=0,\qquad \chi_1=3.26944,$$
$$\chi_2=5.35858,\quad \chi_3=7.39742,$$
$$\chi_4=9.41943.$$

When χ is imaginary, the only admissible root of the transcendental frequency equation is $\chi=0$, which can easily be ascertained from the monotonic behavior of the modified Bessel functions. This confirms our aforementioned conjecture that there are no admissible forms of solutions in the range $\Omega < \xi$.

Periodically structured cylinder

Consider now the case of a composite cylinder with periodic structure in which the lattice distance of the unit cell is $d \equiv (l + l')$. For plane harmonic waves propagating in the z-direction, the tangential component of the displacement in each lamina of the unit cell can be expressed in the form

$$u_\theta = J_1\left(\frac{\pi}{2a}\chi r\right)\left[A \exp\frac{i\pi}{2a}\xi z + B \exp\frac{-i\pi}{2a}\xi z\right]\exp i\omega t, \qquad 0 \le z \le l$$

$$u'_\theta = J_1\left(\frac{\pi}{2a}\chi r\right)\left[A' \exp\frac{i\pi}{2a}\xi' z + B' \exp\frac{-i\pi}{2a}\xi' z\right]\exp i\omega t, \quad -l' \le z \le 0 \tag{17}$$

where ω is the common frequency, since the two laminae are bonded at their common interface, comprising one unit cell. The two wave-numbers ξ and ξ' in the two laminae are given by

$$\xi = (\Omega^2 - \chi^2)^{1/2},$$
$$\xi' = [(\sigma\Omega)^2 - \chi^2]^{1/2}, \tag{18}$$

where

$$\sigma = c/c',$$
$$c = (\mu/\rho)^{1/2}, \quad c' = (\mu'/\rho')^{1/2}.$$

Here c and c' are, respectively, the wave speeds of the shear wave in the two media, and χ are the simple zeros of the transcendental eqn (16).

For the problem of the laminated cylinder with periodic structure of real period d, the solution (17) must satisfy the continuity conditions at the interface $z = 0$. In addition, at the interface $z = l$, the solution must also satisfy the quasi-periodic continuity conditions, since for periodic structure, Floquet's theorem asserts that the amplitude of the wave has the same period as that of the periodic medium. Thus, for time-harmonic waves of the form

$$u_\theta(r, z, t) = u_\theta(r, z)\exp i\omega t, \tag{19}$$

Floquet's theorem[6] asserts that there are solutions of the differential eqn (10) of the form

$$u_\theta(r, z) = w(r, z)\exp\frac{i\pi}{2a}\lambda z, \tag{20}$$

where $w(r, z) = w(r, z + d)$ is a regular periodic function of real period d and λ is a wave-number to be determined by the governing differential operator. The wave-number λ can be real, imaginary or complex. Real values of λ give rise to propagating waves, complex values of λ give rise to attenuated waves, and thus the spectrum consists of passing and stopping bands. When $\lambda = 0$, the solution is periodic with period d, and when $\lambda = 2a/d$, the solution is periodic with period $2d$. Such solutions for $u_\theta(r, z)$ are known as Floquet solutions and can be evaluated over a unit cell by using the continuity condition at $z = 0$ and the quasi-periodic condition at the interface $z = l$.

From these considerations it follows that the general solution (17) of the differential eqn (11) must satisfy the following four conditions

(i) interface continuity condition at $z = 0$

$$u_\theta(r, 0) = u'_\theta(r, 0),$$
$$\tau_{z\theta}(r, 0) = \tau'_{z\theta}(r, 0). \tag{21$_1$}$$

(ii) quasi-periodicity condition at $z = l$

$$u_\theta(r, l) = u'_\theta(r, -l')\exp\frac{i\pi}{2a}\lambda d.$$

$$\tau_{z\theta}(r, l) = \tau'_{z\theta}(r, -l')\exp\frac{i\pi}{2a}\lambda d. \tag{21$_2$}$$

It follows from (20) that Floquet's solutions for all values of z satisfy the quasi-periodic recurrence relation

$$u_\theta(r, z+d) = u_\theta(r, z)\exp\frac{i\pi}{2a}\lambda d. \tag{22}$$

In this equation if we replace λ by λ_0 given by the equation

$$\lambda_0 = \lambda + 4na/d, \quad n = 1, 2, 3, \ldots \tag{23}$$

where $4a/d$ is the length of the reciprocal lattice, then the equation remains unchanged since $\exp 2in\pi = 1$. Therefore in any eigenfunction solution the wave-number λ_0 is exactly equivalent to λ, and is not uniquely defined. We may therefore restrict the values of λ to a unit cell of the reciprocal lattice, since any other value obtained by lattice translation, is exactly equivalent. In the case of propagating waves, it is, however, more convenient to restrict λ to the range $-2a/d \le \lambda \le 2a/d$, rather than $0 \le \lambda \le 4a/d$. Furthermore, the differential operator along with its bilinear concomitant constitutes a self-adjoint system, and therefore it is sufficient to restrict λ to the interval $0 \le \lambda \le 2a/d$, [7]. In one-dimensional reciprocal space $\lambda = 2a/d$ is the boundary of the first Brillouin zone [8].

With this restriction on the values of λ, the dispersion spectrum has a band structure and consists of passing bands which transmit Floquet waves, and stopping bands which do not transmit such waves. As we shall see, for frequencies located in the stopping band the modes decay exponentially since the wave-number λ is in general complex-valued.

To obtain the dispersion spectrum we need the relation between the frequency Ω and wave-number λ. Thus, substituting the general solution (17) into the continuity and quasi-periodicity conditions (21), after making use of the stress-strain relation $(8)_4$, we obtain a system of four linear homogeneous equations in five unknowns. For non-trivial solutions the determinant of the coefficients must vanish and this leads to the characteristic frequency equation

$$\Delta(\Omega, \lambda) \equiv \begin{vmatrix} 1 & 1 & 1 & 1 \\ \mu\xi & -\mu\xi & \mu'\xi' & -\mu'\xi' \\ e^{i\theta} & e^{-i\theta} & e^{-i(\theta'-\phi)} & e^{i(\theta'+\phi)} \\ \mu\xi\, e^{i\theta} & -\mu\xi\, e^{-i\theta} & \mu'\xi'\, e^{-i(\theta'-\phi)} & -\mu'\xi'\, e^{i(\theta'+\phi)} \end{vmatrix} = 0, \tag{24}$$

where

$$\theta \equiv \frac{\pi l}{2a}\xi, \quad \theta' \equiv \frac{\pi l'}{2a}\xi', \quad \phi \equiv \frac{\pi d}{2a}\lambda.$$

The determinant $\Delta(\Omega, \lambda)$ is real and using Laplace's expansion theorem the frequency equation can be written in the convenient form

$$4\mu\mu'\xi\xi'\cos\frac{\pi}{2}(t+t')\lambda + (\mu\xi - \mu'\xi')^2\cos\frac{\pi}{2}(t\xi - t'\xi') - (\mu\xi + \mu'\xi')^2\cos\frac{\pi}{2}(t\xi + t'\xi') = 0, \tag{25}$$

where $t \equiv l/a$ and $t' = l'/a$. We also see from here that $\Delta(\Omega, \lambda)$ is an *even* periodic function of λ with period $4a/d$. For traveling waves we may therefore restrict λ to $-2a/d \le \lambda \le 2a/d$). In this zone Ω is a multiple-valued function of λ, and we call it the reduced zone representation. Alternatively, if we use all real values of λ from $-\infty$ to ∞, we are led to the extended zone representation, and in this representation Ω is, in general, a discontinuous function of λ. The discontinuities in the extended zone scheme occur at $\lambda = \pm 2an/d$ and since $\Delta(\Omega, \lambda)$ is an even periodic function of λ, the slope $\partial\Omega/\partial\lambda$ is zero at these points. There are, however, exceptional cases when $\partial\Delta/\partial\Omega$ and $\partial\Delta/\partial\lambda$ both vanish simultaneously and the slope $\partial\Omega/\partial\lambda$ is non-zero at the end of the zone. These are the critical points of the frequency equation with *non-simple* zeros, where the stopping band vanishes, giving rise to coalescence of *cut-off* frequencies.

We may also note that this frequency equation can also be written in one of the following two equivalent forms

$$\text{(i)}\ \ \mu\mu'\xi\xi'\cos^2\frac{\pi}{4}(t+t')\lambda - \left(\mu'\xi'\cos\frac{\pi}{4}t\xi\cos\frac{\pi}{4}t'\xi' - \mu\xi\sin\frac{\pi}{4}t\xi\sin\frac{\pi}{4}t'\xi'\right)$$

$$+\left(\mu\xi\cos\frac{\pi}{4}t\xi\cos\frac{\pi}{4}t'\xi' - \mu'\xi'\sin\frac{\pi}{4}t\xi\sin\frac{\pi}{4}t'\xi'\right) = 0,$$

$$\text{(ii)}\ \ \mu\mu'\xi\xi'\sin^2\frac{\pi}{4}(t+t')\lambda - \left(\mu'\xi'\cos\frac{\pi}{4}t\xi\sin\frac{\pi}{4}t'\xi' + \mu\xi\sin\frac{\pi}{4}t\xi\cos\frac{\pi}{4}t'\xi'\right)$$

$$\left(\mu\xi\cos\frac{\pi}{4}t\xi\sin\frac{\pi}{4}t'\xi' + \mu'\xi' + \mu'\xi'\sin\frac{\pi}{4}t\xi\cos\frac{\pi}{4}t'\xi'\right) = 0, \tag{26}$$

which suggests that at each end of the zone, the frequency equation uncouples.

3. FREQUENCY SPECTRUM

General properties

Due to the periodic structure of the laminated cylinder, it is expected that the frequency spectrum, i.e. the family of curves giving the frequency as a function of wave-number will have a band structure consisting of passing and stopping bands of frequencies. In addition, the spectrum in the direction of the abscissa (i.e. Floquet's wave-number λ) will be periodic with period $4a/d$. For a general discussion of banded spectra, which can be represented graphically using either the *extended zone scheme* $(0 < \lambda < \infty)$ or the *reduced zone scheme* $(0 < \lambda < 2a/d)$, reference is made to Brillouin[8]. A typical frequency spectrum plotted on the extended zone scheme, showing real, imaginary and complex branches, is displayed in Fig. 2. Solid (broken) lines indicate segments of branches with positive (negative) group velocity, representing a spectrum for waves (energy) moving in the positive (negative) z-direction. Shaded regions represent stopping (i.e. forbidden)

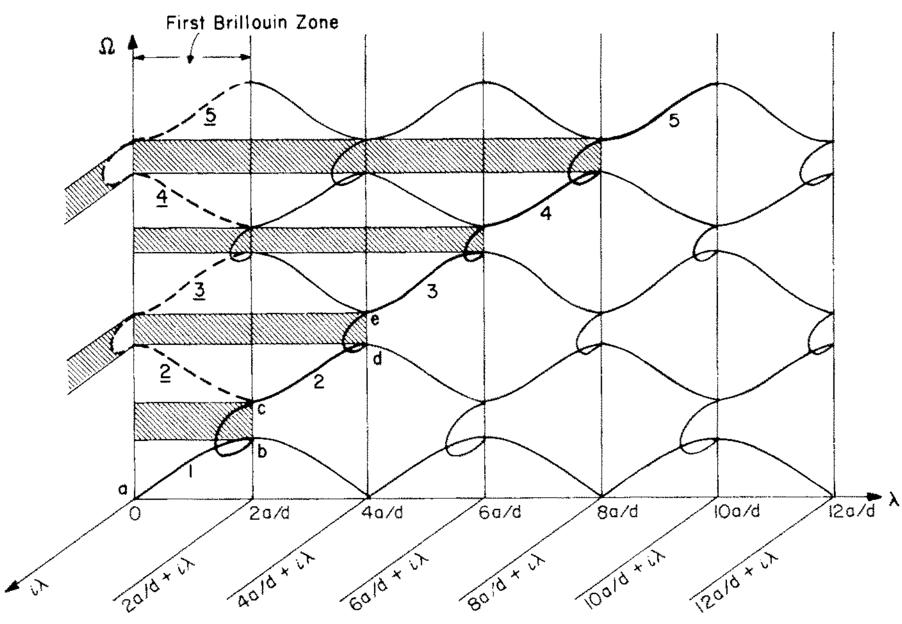

Fig. 2. A typical dispersion spectrum plotted on an extended zone scheme, showing real and imaginary branches. On the reduced zone scheme the same spectrum is shown as branches 1, 2, 3, ... On the extended zone scheme group velocity is positive on each of the segments 1, 2, 3, ... Width of Brillouin zone $2a/d$.

bands with imaginary or complex wave-numbers while unshaded regions indicate passing bands with real wave-numbers λ. On the reduced zone scheme, one considers only the first zone $0 \leq \lambda \leq 2a/d$ and use is made of the suffixes $n = 1, 2, 3, \ldots$ to identify the bands, and the wave-number $\lambda = \lambda_n + 2(n-1)a/d$.

At the two end-points of the zone, corresponding to limiting frequencies between the passing bands of frequencies with real wave-numbers and the stopping bands of frequencies associated with imaginary or complex wave-numbers, $\pi(t+t')\lambda = 0$ or 2π. The mode shapes in the limiting cases take a particularly simple form, because the frequency eqn (25) now uncouples, at each end of the zone, into two frequency equations, as can be seen from eqns (26). Thus from eqn $(26)_2$ we see that when $\lambda = 0$, the two uncoupled frequency equations are

$$\mu\xi \tan\frac{\pi}{4} t\xi + \mu'\xi' \tan\frac{\pi}{4} t'\xi' = 0,$$
$$\mu\xi \tan\frac{\pi}{4} t'\xi' + \mu'\xi' \tan\frac{\pi}{4} t\xi = 0. \tag{27}$$

At the other end of the zone, when $\lambda = 2/(t+t')$, we obtain from eqn $(26)_1$, the two uncoupled frequency equations

$$\mu\xi \tan\frac{\pi}{4} t\xi - \mu'\xi' \cot\frac{\pi}{4} t'\xi' = 0,$$
$$\mu\xi \cot\frac{\pi}{4} t\xi - \mu'\xi' \tan\frac{\pi}{4} t'\xi' = 0, \tag{28}$$

where ξ and ξ' are defined by $(18)_{1,2}$. These uncoupled forms of the frequency equations are valid for all values of radial eigenvalues χ_n, $n \geq 0$.

In the case of torsional modes, the radial eigenvalue $\chi_0 = 0$ is the lowest root of the transcendental frequency eqn (16) and the turning point solution of the differential eqn (11) has a simple form $(14)_2$, which represents a linear variation of the tangential displacement along the radius of the cylinder. In this simple case, for the quasi-periodic torsional waves propagating normal to the layers with periodic structure, the characteristic Floquet's frequency eqn (25) takes the form

$$4\sigma \cos\frac{\pi}{2}(t+t')\lambda + \left(\sqrt{\left(\frac{\mu}{\mu'}\right)} - \sigma\sqrt{\left(\frac{\mu'}{\mu}\right)}\right)^2 \cos\frac{\pi}{2}(t-\sigma t')\Omega$$
$$- \left(\sqrt{\left(\frac{\mu}{\mu'}\right)} + \sigma\sqrt{\left(\frac{\mu'}{\mu}\right)}\right)^2 \cos\frac{\pi}{2}(t+\sigma t')\Omega = 0, \tag{29}$$

where $\sigma = [(\mu/\mu')(\rho'/\rho)]^{1/2}$, and λ is the characteristic exponent of the Floquet waves.

Just as before, eqn (29) can be written in one of the following two equivalent forms

$$\text{(i)}\quad \sigma\mu\mu' \cos^2\frac{\pi}{4}(t+t')\lambda - \left(\sigma\mu' \cos\frac{\pi}{4} t\Omega \cos\frac{\pi}{4}\sigma t'\Omega - \mu \sin\frac{\pi}{4} t\Omega \sin\frac{\pi}{4}\sigma t'\Omega\right)$$
$$\left(\mu \cos\frac{\pi}{4} + \mu \sin\frac{\pi}{4} t\Omega \cos\frac{\pi}{4}\sigma t'\Omega\right)$$
$$\left(\mu \cos\frac{\pi}{4} t\Omega \sin\frac{\pi}{4}\sigma t'\Omega + \sigma t'\Omega - \sigma\mu' \sin\frac{\pi}{4} t\Omega \sin\frac{\pi}{4}\sigma t'\Omega\right) = 0.$$

$$\text{(ii)}\quad \sigma_{\mu\mu'} \sin^2\frac{\pi}{4}(t+t')\lambda - \left(\sigma\mu' \cos\frac{\pi}{4} t\Omega \sin\frac{\pi}{4}\sigma t'\Omega + \mu \sin\frac{\pi}{4} t\Omega \cos\frac{\pi}{4}\sigma t'\Omega\right)$$
$$\left(\mu \cos\frac{\pi}{4} t\Omega \sin\frac{\pi}{4}\sigma t'\Omega + \sigma\mu' \sin\frac{\pi}{4} t\Omega \cos\frac{\pi}{4}\sigma t'\Omega\right) = 0. \tag{29a}$$

At each of the two end-points of the zone, the characteristic Floquet eqns (29a) again uncouple, and when $\lambda = 0$, we get

$$\mu\left(\tan\frac{\pi}{4} t\Omega\right)^{\pm 1} + \sigma\mu'\left(\tan\frac{\pi}{4}\sigma t'\Omega\right)^{\pm 1} = 0, \tag{30$_1$}$$

and when $\lambda = 2/(t+t')$, we get

$$\mu\left(\tan\frac{\pi}{4}t\Omega\right)^{\pm 1} - \sigma\mu'\left(\cot\frac{\pi}{4}\sigma t'\Omega\right)^{\pm 1} = 0. \qquad (30)_2$$

In the passing bands the wave-number λ is real and the dispersion spectrum for real λ is completely described by the characteristic eqn (29) and the cut-off frequencies at the end-points of the zone are given by eqns (30). In the stopping band λ is either imaginary or complex, depending upon the left-end or right-end-side of the zone, respectively. At the left-end-point of the zone, the (Ω, λ)-spectrum for imaginary wave-numbers is governed by the equation

$$4\sigma\epsilon_{(\lambda)}\cosh\frac{\pi}{2}(t+t')\lambda + \left(\sqrt{\frac{\mu}{\mu'}} - \sigma\sqrt{\frac{\mu'}{\mu}}\right)^2 \cos\frac{\pi}{2}(t-\sigma t')\Omega - \left(\sqrt{\frac{\mu}{\mu'}} + \sigma\sqrt{\frac{\mu}{\mu}}\right)^2 \cos\frac{\pi}{2}(t+\sigma t')\Omega = 0, \qquad (31)$$

where $\epsilon_{(\lambda)} = 1$. At the right end-point of the zone, when the segment of the spectrum is complex, the (Ω, λ)-branch for complex wave-numbers is again governed by eqn (31), but now $\epsilon_{(\lambda)} = -1$. The complex branch is again a plane curve lying completely in the shifted $(\Omega, 2a/d + i\lambda)$-plane. The frequency spectrum for the pure torsional mode is shown in Fig. 3.

In the general case when $\chi_n > 0$, the imaginary and complex branches at the two end-points of the banded spectrum are governed by the transcendental frequency equation

$$4\mu\mu'\xi\xi'\epsilon_{(\lambda)}\cosh\frac{\pi}{2}(t+t')\lambda + (\mu\xi - \mu'\xi')^2\cos\frac{\pi}{2}(t\xi - t'\xi') - (\mu\xi + \mu'\xi')^2\cos\frac{\pi}{2}(t\xi + t'\xi') = 0. \qquad (32)$$

At the left end-point when $\epsilon_{(\lambda)} = +1$, the curve lies in the $(\Omega, i\lambda)$-plane and at the right end-point of the zone where $\epsilon_{(\lambda)} = -1$, the curve lies in the $(\Omega, 2a/d + i\lambda)$-plane.

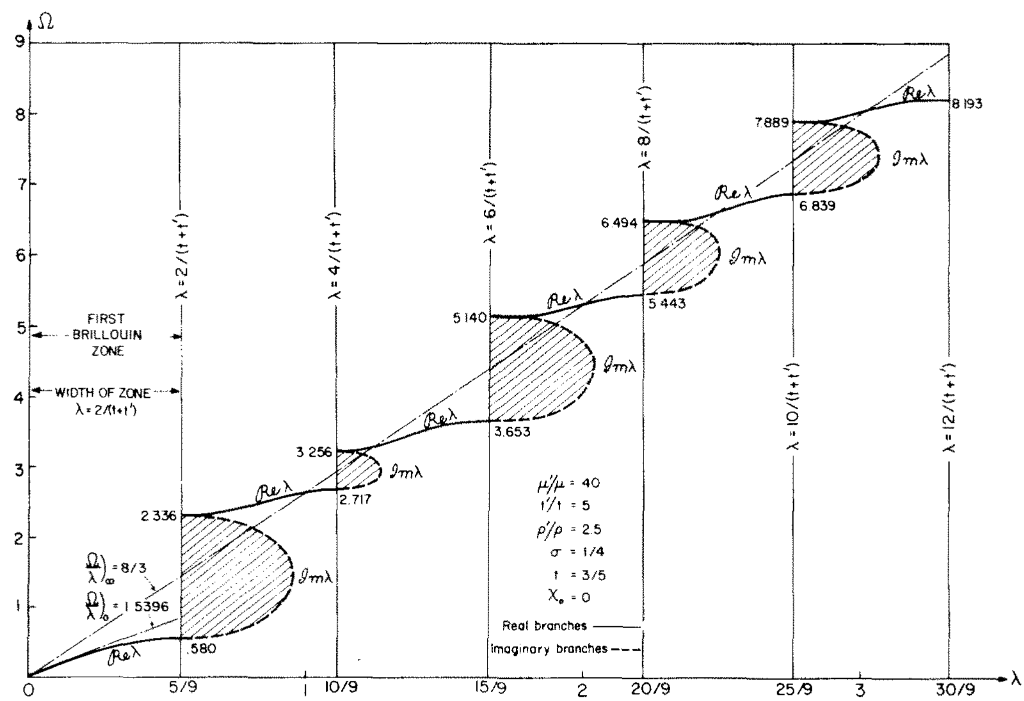

Fig. 3. Frequency spectrum for pure torsional modes, $\chi = 0$, shown on an extended zone scheme for first six Brillouin zones. Imaginary segments of the branches are shown as dotted lines, real segments are shown as full lines. Stopping bands are hatched regions, passing bands are unhatched. At long wave-length limit $[\Omega/\lambda]_0 = 1.5396$, and at short wave length limit $[\Omega/\lambda]_\infty = (t+t')/(t+\sigma t') = 8/3$.

In the zero-frequency plane $\Omega = 0$ and therefore from $(18)_{1,2}$, the wave-numbers ξ and ξ' are both imaginary. Thus when $\Omega = 0$

$$\xi = i\chi, \quad \xi' = i\chi \tag{33}$$

and the frequency equation, governing the origin of complex branches on the $\mathscr{I}m\lambda$-axis in the zero frequency plane are

$$\text{(i)}\ \sinh^2\frac{\pi}{4}(t+t')\lambda = \left(\sqrt{\frac{\mu}{\mu'}}\cosh\frac{\pi}{4}t\chi\sinh\frac{\pi}{4}t'\chi + \sqrt{\frac{\mu'}{\mu}}\sinh\frac{\pi}{4}t\chi\cosh\frac{\pi}{4}t'\chi\right)$$
$$\left(\sqrt{\frac{\mu'}{\mu}}\cosh\frac{\pi}{4}t\chi\sinh\frac{\pi}{4}t'\chi + \sqrt{\frac{\mu}{\mu'}}\sinh\frac{\pi}{4}t\chi\cosh\frac{\pi}{4}t'\chi\right),$$
$$\text{(ii)}\ \cosh^2\frac{\pi}{4}(t+t')\lambda = \left(\sqrt{\frac{\mu}{\mu'}}\cosh\frac{\pi}{4}t\chi\cosh\frac{\pi}{4}t'\chi + \sqrt{\frac{\mu'}{\mu}}\sinh\frac{\pi}{4}t\chi\sinh\frac{\pi}{4}t'\chi\right)$$
$$\left(\sqrt{\frac{\mu'}{\mu}}\cosh\frac{\pi}{4}t\chi\cosh\frac{\pi}{4}t'\chi + \sqrt{\frac{\mu}{\mu'}}\sinh\frac{\pi}{4}t\chi\sinh\frac{\pi}{4}t'\chi\right). \tag{34}$$

For $\chi > 0$, the simple zeros of eqn $(34)_1$ lie on the $(0, \mathscr{I}m\lambda)$-axis and the zeros of eqn $(34)_2$ lie on the $(0, 2a/d + \mathscr{I}m\lambda)$-axis. These points are the origin of complex branches on the zero frequency plane. For pure torsional modes $\chi_0 = 0$, and real roots of these equations are given by

$$\cos\frac{\pi}{2}(t+t')\lambda = 1,$$
$$\lambda = 4n/(t+t'), \quad n = 0, 1, 2, \ldots \tag{35}$$

Therefore inside the first zone, including the end-points $0 \le \lambda \le 2/(t+t')$, the only root on the $(0, \lambda)$-axis is the origin $(0, 0)$. For $\chi > 0$, there are no roots of the eqn $(34)_1$ on the $(0, \mathscr{R}e\lambda)$-axis, but there is a root on the $(0, \mathscr{I}m\lambda)$-axis. In addition, from the propertis of hyperbolic functions it can be seen that there are no roots of the eqn (34) on the $(0, 2/(t+t') + \mathscr{I}m\lambda)$-axis. For $\chi_1 = 3.26944$ and for the choice of parameters used in this paper, the root on the $(0, \mathscr{I}m\lambda)$-axis is shown in Fig. 4 with coordinates $\Omega = 0$, $\mathscr{I}m\lambda = 3.684335$.

Transitional cases

In our previous discussion we have already considered one transitional (degenerate) form of the solution when $\chi_0 = 0$. In this case the motion is a pure torsional mode and the band spectrum for the Floquet waves is shown in Fig. 3. In this problem other transitional cases arise, when

$$\begin{aligned}&\text{(A)}\ \xi = 0, \quad \text{i.e. when } \Omega = \chi, \quad \text{and } \xi' \neq 0\\ &\text{(B)}\ \xi' = 0, \quad \text{i.e. when } \Omega = \chi/\sigma, \quad \text{and } \xi \neq 0\\ &\text{(C)}\ \xi = \xi' = 0, \quad \text{i.e. when } \sigma = 1.\end{aligned} \tag{36}$$

Case A. When $\xi = 0$, $\Omega = \chi$ and eqn (25) vanishes identically, which suggests degeneracy. In this case eqn $(17)_1$ is not an appropriate form of the solution of a second order differential equation. However, the characteristic equation in this case can easily be obtained from eqn (25), by first differentiating it with respect to ξ, and then taking the limit of the differentiated form as $\xi \to 0$, that is, when

$$\Delta(\Omega, \lambda) \equiv 0, \qquad \operatorname{Lim} \xi = 0 \tag*{(37)$_1$}$$

the limiting form of the characteristic equation is given by

$$\left[\frac{\partial}{\partial\xi}\Delta(\Omega, \lambda)\right] = 0.$$
$$\operatorname{Lim}\xi = 0 \tag*{(37)$_2$}$$

Thus, following this procedure, we immediately obtain from (25) and (37) the equation

$$\cos\frac{\pi}{2}(t+t')\lambda - \cos\frac{\pi}{2}t'\xi_0' + \frac{1}{4}\pi t\frac{\mu'}{\mu}\xi_0'\sin\frac{\pi}{2}t'\xi_0' = 0, \tag{38}$$

where, when $\xi = 0$, $\Omega = \chi$ and therefore $\xi_0' = (\sigma^2 - 1)^{1/2}\chi$. Substituting in (38), we find that in this case λ are the roots of the equation

$$\cos\frac{\pi}{2}(t+t')\lambda = \cosh\frac{\pi}{2}t'\sqrt{(1-\sigma^2)}\chi + \frac{1}{4}\pi t\frac{\mu'}{\mu}\sqrt{(1-\sigma^2)}\chi\,\sinh\frac{\pi}{2}t'\sqrt{(1-\sigma^2)}\chi. \tag{39}$$

For $\chi_n > 0$ and $\sigma < 1$, this equation has only one root with coordinates ($\Omega = \chi_n$, $\mathscr{I}m\lambda$). For $\Omega = \chi$, such a root on the $\mathscr{I}m\lambda$-axis is shown as point A in Fig. 4. On the other hand, when $\chi = 0$, eqn (39) degenerates to the form (35), discussed previously. We may also note that $\Omega = \chi$, $\lambda = 0$ is a root of eqn (39), if t and t' are so chosen that they satisfy the equation $-\mu\tanh\theta t' = \mu'\theta t$ where $\theta = (1/4)\mu\chi\sqrt{(1-\sigma^2)}$. On the other hand, if t and t' satisfy the equation $-\mu\coth\theta t' = \mu'\theta t$, then the root of the equation is $\lambda = 2/(t+t')$, which is the right end-point of the zone.

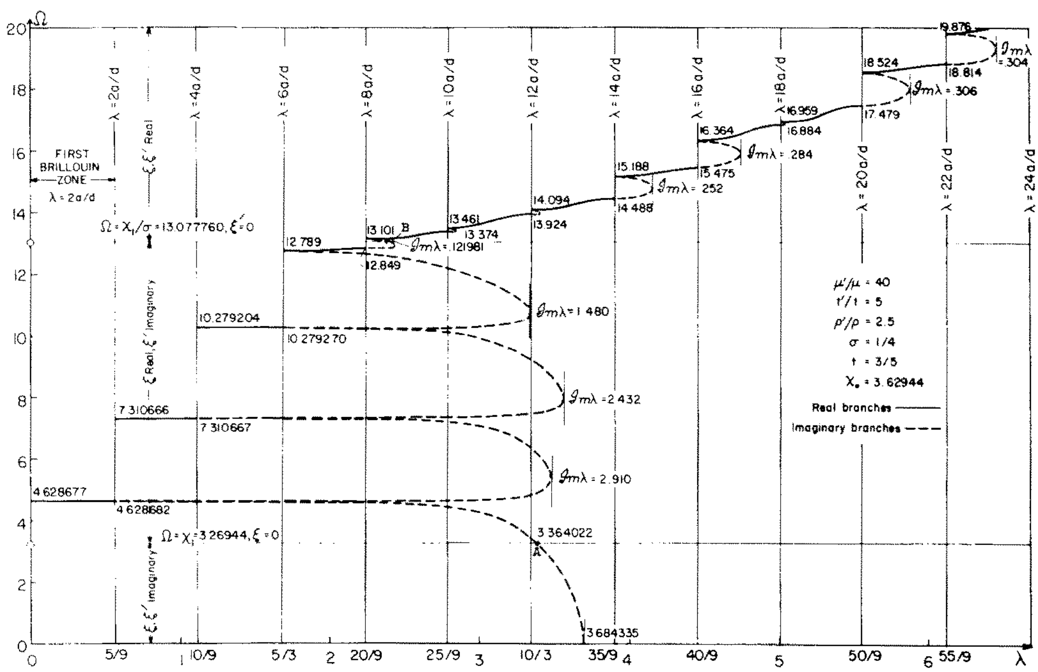

Fig. 4. Frequency spectrum for torsional modes with one radial node, $\chi_1 = 3.62944$, shown on an extended zone scheme for first twelve Brillouin zones. Imaginary segments are shown as dotted lines, real segments are shown as full lines. When $\xi = 0$ we have turning point A, and turning point B corresponds to $\xi' = 0$.

In the region $0 < \Omega < \chi$, the wave-numbers ξ and ξ' are both imaginary when $\sigma < 1$. In particular, in the degenerate case $\xi = 0$, the mode shape is given by

$$u_\theta = (A + Bz)J_1\left(\frac{\pi}{2a}\chi r\right)\exp i\omega t, \qquad 0 < z < l$$

$$u_\theta' = \left(A'\exp\frac{i\pi}{2a}\xi_0' z + B'\exp\frac{-i\pi}{2a}\xi_0' z\right)J_1\left(\frac{\pi}{2a}\chi r\right)\exp i\omega t, \qquad -l' < z < 0 \tag{40}$$

when $\Omega = \chi$ and $\xi_0' = (\sigma^2 - 1)^{1/2}\Omega$. Thus in section $0 < z < l$, the tangential displacement u_θ has a linear variation along the length of the cylinder, and in section $-l' < z < 0$, the tangential displacement u_θ' has hyperbolic variation.

Case B. When $\xi' = 0$, $\Omega = \chi/\sigma$ and eqn (25) again vanishes identically. The mode shape $(17)_2$ in this case is no longer valid and we have to consider instead a degenerate form of the solution.

Since eqn (25) vanishes identically in this case, the appropriate frequency equation is given by

$$\left[\frac{\partial}{\partial \xi'} \Delta(\Omega, \lambda)\right]_{\lim \xi' = 0} = 0. \tag{41}$$

Therefore from (25) and (41) we obtain the limiting form

$$\cos\frac{\pi}{2}(t+t')\lambda = \cos\frac{\pi t}{2\sigma}\chi\sqrt{(1-\sigma^2)} - \frac{\pi t'}{4\sigma}\frac{\mu}{\mu'}\chi\sqrt{(1-\sigma^2)}\sin\frac{\pi t}{2\sigma}\chi\sqrt{(1-\sigma^2)} = 0, \tag{42}$$

for $\sigma < 1$. When $\xi' = 0$, $\Omega = \chi/\sigma$ and therefore $\xi_0 = (1-\sigma^2)^{1/2}(\chi/\sigma)$ is real. For given values of the parameters and a given χ, this equation determines the admissible values of the wave-number λ. One such root is shown in Fig. 4 as point B. Some particularly simple solution of this equation can easily be obtained for suitably chosen values of t and t'.

$$\text{(i) If} \quad t = \frac{2\sigma}{\chi\sqrt{(1-\sigma^2)}} \quad \text{then} \quad \lambda = 2/(t+t'). \tag{43}$$

In this case, on the $\Omega = \chi/\sigma$ line, the root lies on the right end-point of the zone.

$$\text{(ii) If} \quad t = \frac{(2n+1)\sigma}{\chi\sqrt{(1-\sigma^2)}}, \quad n = 0, 1, 2, \ldots$$

then

$$\cos\frac{\pi}{2}(t+t')\lambda = (-1)^{n+1}(2n+1)\frac{\pi}{4}\left(\frac{t'/t}{\mu'/\mu}\right), \tag{44}$$

and the value of λ may now be real, imaginary or complex.

$$\text{(iii) If} \quad t' = \frac{4}{\pi}\frac{\mu'}{\mu}\frac{\sigma}{\chi\sqrt{(1-\sigma^2)}}\cot\frac{\pi t}{2\sigma}\chi\sqrt{(1-\sigma^2)}, \quad \text{then} \quad \lambda = 1/(t+t').$$

$$\text{(iv) If} \quad t = \frac{\sigma}{2\chi\sqrt{(1-\sigma^2)}}, \quad t' = \frac{4\sigma\mu'}{\pi\mu\chi\sqrt{(1-\sigma^2)}}, \quad \text{then} \quad \lambda = 1/(t+t'). \tag{45}$$

On the line $\Omega = \chi/\sigma$, the wave-number $\xi' = 0$ and ξ is real. In the region $\chi \leq \Omega < \chi/\sigma$, the wave-number ξ' is imaginary and ξ remains real. In particular, when $\Omega = \chi/\sigma$, $\xi' = 0$ and the degenerate form of the mode shape at this frequency is given by

$$u_\theta = \left(A\exp\frac{i\pi}{2a}\xi_0 z + B\exp\frac{-i\pi}{2a}\xi_0 z\right)J_1\left(\frac{\pi}{2a}\chi r\right)\exp i\omega t, \quad 0 < z < l$$

$$u'_\theta = (A' + B'z)J_1\left(\frac{\pi}{2a}\chi r\right)\exp i\omega t, \qquad -l' < z < 0 \tag{46}$$

where $\xi_0 = \Omega\sqrt{(1-\sigma^2)}$. Thus, in the section $-l' < z < 0$, the tangential displacement u'_θ varies linearly along the length of the cylinder, and in the adjacent section $0 < z < l$, the tangential displacement u_θ varies sinusoidally.

Case C. Consider now the case when the speed of the shear waves in the two materials is the same. In this case $\sigma = 1$ and consequently $\xi = \xi'$. The frequency eqn (25) now takes the simple form

$$\cos\frac{\pi}{2}(t+t')\lambda - \cos\frac{\pi}{2}t\xi\cos\frac{\pi}{2}t'\xi + \frac{1}{2}\left(\frac{\mu}{\mu'} + \frac{\mu'}{\mu}\right)\sin\frac{\pi}{2}t\xi\sin\frac{\pi}{2}t'\xi = 0. \tag{47}$$

When $\mu = \mu'$ and $\sigma = 1$, we recover the expected result for an infinite, homogeneous cylinder

$$\cos\frac{\pi}{2}(t+t')\lambda = \cos\frac{\pi}{2}(t+t')\xi. \tag{48}$$

Hence, $\lambda \pm 4na/d = \xi$ is a solution of the equation, where n is an integer. For a homogeneous medium the period d is arbitrary, and therefore for the solution to be valid for all arbitrary values of d, it is necessary that $n = 0$. This leads to the solution $\lambda = \xi$. The dispersion spectrum for a homogeneous cylinder is, therefore, given by $\Omega^2 = \lambda^2 + \chi_n^2$. In the (Ω, λ, χ)-space this represents a three-dimensional curve. The locus of the points describing the curve is the intersection of the cone with the plane $\chi_n = \text{const}$. For waves propagating normal to the layers, the dispersion spectrum is a straight line $\Omega = \lambda$. However, when $\mu \neq \mu'$, we have a banded spectrum governed by eqn (47).

In the degenerate case when $\xi = 0$, $\Omega = \chi$ and the value of λ is given by

$$\cos\frac{\pi}{2}(t+t')\lambda = 1. \tag{49}$$

Thus $\lambda = 0, 4\pi a/d, 8\pi a/d, \ldots$ are all meaningful in the sense of eqn (23). The degenerate mode shapes are

$$u_\theta = (A+Bz)J_1\left(\frac{\pi}{2a}\chi r\right)\exp i\omega t, \qquad 0 \le z \le l$$

$$u'_\theta = (A'+B'z)J_1\left(\frac{\pi}{2a}\chi r\right)\exp i\omega t, \qquad -l' \le z \le 0, \tag{50}$$

and in this case the tangential displacements u_θ and u'_θ, both vary linearly along the length of the cylinder.

For $\Omega < \chi$, the wave-number ξ is imaginary, and the mode shapes have hyperbolic variation in each lamina.

Properties for vanishing wave-number

In order to plot the spectrum it is useful to determine the ordinates (cut-off frequencies), slopes and curvatures of the branches at the two end-points of the zone $\lambda = 0$ and $\lambda = 2a/d$. If one knows these quantities it is possible to gain a quantitative idea with regard to the complete frequency spectrum.

In the case of pure torsional modes, when $\chi = 0$, the cut-off frequencies at the end-points of the zone are given by eqns $(30)_{1,2}$ which gives us the ordinate of the branches at the end-points. From eqn $(29)_2$ it is obvious that the origin $\Omega = 0$, $\lambda = 0$ is an exceptional point and the slope of the branch emanating from the origin is finite. Excluding this point, the slope of the branches at the remaining cut-off frequencies, at each of the two ends of the zone is generally zero. This observation follows from the fact that the frequency equation is an even periodic function of λ. From eqn (29) it can also be shown that when the slope is zero, the curvature of the branch is given by

$$\left.\frac{d^2\Omega_n}{d\lambda^2}\right]_{\substack{\lambda=0(2a/d)\\ n\neq 0}} = \frac{2\sigma\pi(d/a)^2\epsilon_{(\lambda)}}{b_1\sin\frac{\pi}{2}(t+\sigma t')\Omega_n - b_2\sin\frac{\pi}{2}(t-\sigma t')\Omega_n}, \tag{51}$$

where Ω_n is the n-th root of the eqn (30), $\epsilon_\lambda = +1(-1)$ depending upon the end-point $\lambda = 0(2a/d)$, and

$$b_1 = (t+\sigma t')(\sqrt{(\mu/\mu')} + \sigma\sqrt{(\mu'/\mu)}),$$
$$b_2 = (t-\sigma t')(\sqrt{(\mu/\mu')} - \sigma\sqrt{(\mu'/\mu)}).$$

In the exceptional case when $\Omega = 0$ is a root of the frequency equation, the slope of the branch $d\Omega/d\lambda \neq 0$ at $\lambda = 0$. From eqn (29) it can be seen that for small values of Ω and λ, the behavior of the branch in the neighborhood of the origin is governed by the equation $d\Omega/d\lambda = \Omega/\lambda$, which suggests that Ω is a linear function of λ. The constant of proportionality can easily be determined by using Taylor expansion in the neighborhood of the origin, and therefore the slope is given by

$$\left.\frac{d^2\Omega_0}{d\lambda}\right]_{\substack{\Omega=0\\ \lambda=0}} = \frac{(t+t')}{\left[\left(t+\frac{\mu}{\mu'}t'\right)\left(t+\sigma^2\frac{\mu'}{\mu}t'\right)\right]^{1/2}}, \tag{52}$$

which is always positive, since the denominator is positive definite. Thus in the first pass-band $n = 1$, the group velocity is always positive and reaches a zero value only at the other end-point of the zone at $\lambda = 2a/d$.

Complex segments of branches

To gain some understanding of the spectrum in the stopping bands, it is necessary to study the region at points of zero group velocity. By using the properties of analytic functions one can show that for complex values of λ and real values of Ω,

$$\Delta(\Omega, \lambda_1 + i\lambda_2) = \Delta(\Omega, \lambda_1) + i\lambda_2 \frac{\partial \Delta}{\partial \lambda_1} = 0, \tag{53}$$

where $\Delta(\Omega, \lambda) = 0$ is the frequency equation. This implies that in the real plane $\lambda_2 = 0$, the dispersion spectrum is given by the frequency equation $\Delta(\Omega, \lambda_1) = 0$. In the neighborhood of the real plane when $\lambda_2 \neq 0$, the complex branches of the spectrum are governed by

$$\Delta(\Omega, \lambda_1) = 0, \qquad \partial\Delta/\partial\lambda_1 = 0. \tag{54}$$

Also $\partial\Delta/\partial\lambda_1 + (\partial\Delta/\partial\Omega)(\partial\Omega/\partial\lambda_1) = 0$, which implies that $\partial\Omega/\partial\lambda_1 = 0$ if $\partial\Delta/\partial\Omega \neq 0$. This indicates that the maximum and minimum of the branches in the real plane are points of intersection, at normal incidence, with a complex segment. It can similarly be shown that the complex branches intersect, at maximum and minimum of the imaginary branches, at normal incidence, provided $\partial\Delta/\partial\Omega \neq 0$. It thus follows that all imaginary and complex branches originate at the end-points of the zone, where the group velocity is zero. The imaginary and complex branches of the spectrum are admissible solutions according to Floquet's theory, but represent non-propagating waves, since the Floquet exponential now represents a wave attenuated exponentially, either to the right or left. Such waves differ only in relative phases of oscillations in the successive (l, l') section of the cylinder, and therefore the frequency band in which λ is complex or imaginary, defines the stopping band, with *unstable* oscillations. On the other hand real values of λ give rise to waves propagating through the whole cylinder without attenuation, which means passing bands and represent *stable* oscillations[6].

Spectrum for radial eigenvalues χ_n, $n \geq 2$.

The spectrum for more than one radial node can be obtained from the spectrum with one radial node by a suitable magnification of scale. This can easily be seen from the fact that in the radial direction u_θ is proportional to $J_1[(\pi/2a)\chi r]$ and the nonvanishing tangential stress $\tau_{r\theta}$ is proportional to $J_2[(\pi/2a)\chi r]$. When $\chi = \chi_1$ and $r = a$, the shear stress $\tau_{r\theta}$ vanishes on the surface of the cylinder. But in this case, if we choose $a = (\chi_1/\chi_2)b$, we can write

$$J_2\left(\frac{\pi}{2a}\chi_1 r\right) = J_2\left(\frac{\pi}{2b}\chi_2 r\right), \tag{55}$$

and this latter expression vanishes on the surface of a cylinder of radius b, provided χ_2 is the second radial eigenvalue. The displacement u_θ in the radial direction is now proportional to $J_1\left(\frac{\pi}{2b}\chi_2 r\right)$ and it has two nodes at $(\pi/2b)\chi_2 r = 3.83171$ and 7.01559, (Ref. [5], p. 441). Since $(\pi/2)\chi_2 = 8.41724$ it follows that the nodes are located at $r/b = 0.45522$ and $r/b = 0.83348$. Thus from the results for a cylinder of radius a and with one node, one can obtain the results for a cylinder of radius b with two nodes, provided the radii of the two cylinders a and b bear the relation $a/b = \chi_1/\chi_2$. In general, one can show that by choosing $a/b = \chi_n/\chi_m$, $m > n$, one can obtain the frequency spectrum for the m-th radial mode from the results of n-th radial mode. Since the χ_0 mode is an exceptional case, it has to be treated separately. It thus follows that having the dispersion spectrum for χ_0 and χ_1, one can obtain the frequency spectrum for χ_2, $\chi_3, \ldots$ etc., from the spectrum of χ_1, by a suitable magnification of scale.

To this end consider the Floquet's characteristic eqn (25) for a cylinder of radius a and one radial node with eigenvalue χ_1. Then this equation has the form $\Delta(\Omega_a, \lambda_a, \chi_1) = 0$, where Ω_a and λ_a are the frequencies and Floquet's characteristic exponent, respectively, for a cylinder of radius a, with one radial node. From this frequency equation we can obtain the frequency equation $\Delta(\Omega_b, \lambda_b, \chi_2) = 0$, if we make the transformations $\Omega_a = (a/b)\Omega_b$, $\lambda_a = (a/b)\lambda_b$, $a/b = \chi_1/\chi_2$. Thus, if we have the (Ω_a, λ_a)-plot for a cylinder of radius a with radial eigenvalue χ_1, we can obtain the (Ω_b, λ_b)-plot for a cylinder of radius b and two radial nodes, provided both the ordinate and the abscissa are magnified by the magnification factor χ_2/χ_1. This process can now be generalized to obtain the spectrum for the higher radial nodes, without repeated computations of the roots of the frequency equations.

Band widths

Some information on the width of the passing and stopping bands can be gained from an examination of the frequency eqns (27) and (28) at the two ends of the zone. Consider first the simpler case of pure torsional nodes. In this case eqn (30) at $\lambda = 0$, indicates that the two frequency equations will have common roots when (a) $\sigma\mu' = \mu$, and (b) $\sigma t' = t$. In the first case eqn $(30)_2$ at the other end of the zone also has a sequence of common roots. This suggests, that in this case there are no branches of the spectrum in the imaginary plane at either end of the zone, and therefore there are no stopping bands. The frequency spectrum in this case, therefore, does not have jumps at the end of the zones and there are no attenuating modes. On the reduced zone scheme, the frequency spectrum is a continuous band, with cut-off frequencies $\Omega = 4n(t + \sigma t')$ at the left-end of the band and $\Omega = 2(2n + 1)/(t + \sigma t')$ at the right-end of the band, $n = 0, 1, 3, \ldots$

In the second case when $\sigma t' = t$, the roots of the two eqns $(30)_1$ at the left end of the zone are the same and are given by $\Omega = 2(2n + 1)/t$. On the right end side of the band the two frequency eqns $(30)_2$, take the form

$$\tan\frac{\pi}{4}t\Omega = \pm\sqrt{(\sigma\mu'/\mu)}, \qquad \cot\frac{\pi}{4}t\Omega = \pm\sqrt{(\sigma\mu'/\mu)}. \tag{56}$$

The roots of these two equations differ from each other by $\pm 2/t$, which is the width of the stopping band, because in this region the wave-number is complex and represents attenuation of the wave. At this end of the zone there is, therefore, a frequency jump and consequently an imaginary branch, but on the left end of the zone there are no imaginary branches and no discontinuity in the frequency. On the reduced zone scheme the frequency spectrum consists of a real band $n = 1$, followed by an imaginary loop, then two real bands $n = 2, 3$ with a common cut-off frequency at $\lambda = 0$, followed again by an imaginary loop at $\lambda = 2a/d$. This process of two real bands followed by an imaginary loop then repeats for higher bands. In addition, it can be shown that the degenerate bands at common cut-off frequencies have non-zero group velocity.

We thus see that in the case of pure torsional modes, there are no stopping bands if $\sigma\mu' = \mu$, that is when the material constants are so chosen that $c/c' = \mu/\mu'$. On the other hand, if the dimensions of the laminate and the material constants are in agreement with the travel-time relation $c'/t' = c/t$, then the width of the stopping band is $\pm 2/t$ and occurs at the end points of the odd numbered zones. In every other case there is a passing band followed by a stopping band in every zone, and this pattern repeats from zone to zone.

A similar analysis can also be carried out in the general case of non-zero radial eigenvalue χ_n. There are no stopping bands when $\mu = \mu'$ and $\sigma = 1$, and this corresponds to the case of the homogeneous cylinder. When $t = t'$ and $\sigma = 1$, there are stopping bands, but only at the right end-points of the odd numbered zones.

4. MODE SHAPES AT ZONE ENDS

At the two ends of the zone where $\lambda = 0$ and $\lambda = 2a/d$, the general Floquet's frequency eqn (26) decomposes into simpler frequency eqns (27) and (28). This, therefore, suggests that at each of the two end-points of the zone and mode shapes (17) must take simpler form. In fact from the recurrence formula (22), one can see that for $n = 0, 1, 2, \dots$, etc., $u_\theta(r, z + nd) = u_\theta(r, z)$ when $\lambda = 0$, and $u_\theta(r, z + nd) = (-1)^n u_\theta(r, z)$ when $\lambda = 2a/d$. Thus in the first case the eigenfunction is periodic with period d and in the second case the period is $2d$.

Consider first the case when $\lambda = 0$. Floquet's condition (21) now requires that at $z = l$

$$u_\theta(r, l) = u'_\theta(r, -l'),$$
$$\mu \partial_z u_\theta(r, l) = \mu' \partial_z u'_\theta(r, -l'), \tag{57}$$

and the continuity of displacement and stress at the interface $z = 0$ requires that

$$u_\theta(r, 0) = u'_\theta(r, 0),$$
$$\mu \partial_z u_\theta(r, 0) = \mu' \partial_z u'_\theta(r, 0). \tag{58}$$

The eigenfunctions (17) can be written as the sum of *even* and *odd* functions. Let $u_\theta(r, z)$ be an even function about $z = l/2$ in $0 \le z \le l$, so that $u_\theta(r, 0) = u_\theta(r, l)$. From $(57)_1$ this can be written as $u_\theta(r, 0) = u'_\theta(r, -l')$. Combining it with $(58)_1$ we get $u'_\theta(r, 0) = u'_\theta(r, -l')$, which implies that $u'_\theta(r, z)$ is also an even function about $z = -l'/2$ in $-l' < z < 0$. When $u_\theta(r, z)$ and $u'_\theta(r, z)$ are both even functions, the stress continuity conditions $(57)_2$ and $(58)_2$ are equivalent. The appropriate eigenfunctions in this case are

$$u_\theta(r, z) = AV(r) \cos \frac{\pi}{2a} \xi(z - l/2), \qquad 0 \le z \le l$$
$$u'_\theta(r, z) = A'V(r) \cos \frac{\pi}{2a} \xi'(z + l'/2), \qquad -l' \le z \le 0 \tag{59}$$

which are even functions with respect to the middle plane of each lamina. We call this combination an *even-even* eigenfunction and from cell to cell it is periodic with period d. The eigenfrequencies in this case are governed by the frequency eqn $(27)_1$.

It can similarly be shown that when $u_\theta(r, z)$ and $u'_\theta(r, z)$ are both odd functions, the continuity conditions (57) and (58) are equivalent. The appropriate eigenfunctions in this case are

$$u_\theta(r, z) = BV(r) \sin \frac{\pi}{2a} \xi(z - l/2), \qquad 0 \le z \le l$$
$$u'_\theta(r, z) = B'V(r) \sin \frac{\pi}{2a} \xi'(z + l'/2), \qquad -l' \le z \le 0 \tag{60}$$

which are both odd functions with respect to the middle plane of each lamina. We call this combination an *odd-odd* eigenfunction and from cell to cell it is periodic with period d. The eigenfrequencies in this case are governed by the frequency eqn $(27)_2$.

Consider now the other end of the zone $\lambda = 2a/d$. Quasi-periodicity condition at $z = l$ now reduces to

$$u_\theta(r, l) = -u'_\theta(r, -l'),$$
$$\mu \partial_z u_\theta(r, l) = -\mu' \partial_z u'_\theta(r, -l'). \tag{61}$$

In addition, the displacement and stress fields must also satisfy the continuity conditions (58) at the interface $z = 0$.

Let $u_\theta(r, z)$ be now an odd function about $z = l/2$ in $0 \le z \le l$. From the properties of odd functions and $(61)_1$ we can write $u_\theta(r, 0) = u'_\theta(r, -l')$. Combining it with $(58)_1$ we get $u'_\theta(r, 0) = u'_\theta(r, -l')$, which implies that $u'_\theta(r, z)$ is an even function about $z = -l'/2$ in $-l' \le z \le 0$. When $u_\theta(r, z)$ is odd and $u'_\theta(r, z)$ is even, the stress continuity condition $(61)_2$ is equivalent to $(58)_2$. The appropriate eigenfunctions in this case are

$$u_\theta(r, z) = BV(r) \sin \frac{\pi}{2a} \xi(z - l/2), \qquad 0 \le z \le l$$
$$u'_\theta(r, z) = A'V(r) \cos \frac{\pi}{2a} \xi'(z + l'/2), \qquad -l \le z \le 0 \tag{62}$$

where $u_\theta(r, z)$ is odd about $z = l/2$ and $u'_\theta(r, z)$ is even about $z = -l'/2$. We call this combination an *odd-even* eigenfunction and from cell to cell it has half-period d. The eigenfrequencies in this case are governed by the frequency eqn $(28)_2$.

It can similarily be shown that the continuity conditions (58) and (61) are equivalent when $u_\theta(r, z)$ is even and $u'_\theta(r, z)$ an odd function. The appropriate eigenfunctions in this case are

$$u_\theta(r, z) = AV(r)\cos\frac{\pi}{2a}\xi(z - l/2), \qquad 0 \le z \le l$$

$$u'_\theta(r, z) = B'V(r)\sin\frac{\pi}{2a}\xi'(z + l'/2), \qquad -l' \le z \le 0. \qquad (63)$$

We call this combination an *even-odd* eigenfunction and from cell to cell it has half-period d. The eigenfrequencies in this case are governed by frequency eqn $(28)_1$.

Typical mode shapes corresponding to these eigenfunctions for pure torsional waves are shown in Fig. 5. Thus $\Omega_0 = 0$ is the lowest root of frequency eqn $(30)_1^+$ and the eigenfunction (59)

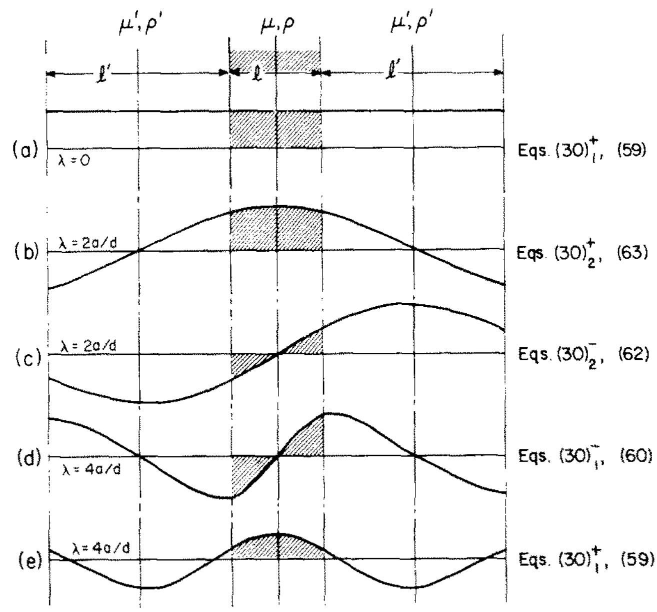

Fig. 5. Mode shapes for pure torsional waves at end-points of the zones when $\lambda = 0$, $2a/d$ and $4a/d$, corresponding to typical cut-off frequencies marked a, b, c, d and e in Fig. 2. Equation numbers applicable in each case are also listed with the mode shapes.

shown in Fig. 5a corresponds to rigid motion. In increasing order, the next cut-off frequency is the lowest root of the frequency eqn $(30)_2^+$ and the corresponding mode shape (63) is shown in Fig. 5b. The next cut-off frequency at the right end-point of the zone is given by eqn $(30)_2^-$ and the corresponding mode shape (62) is shown in Fig. 5c. On the extended zone scheme the next set of cut-off frequencies corresponds to the end of the zone when $\lambda = 4a/d$. On the reduced zone scheme this corresponds to the case $\lambda = 0$. In increasing order the next higher frequency is the root of eqn $(30)_1^-$ and the corresponding eigenfunction (60) is shown in Fig. 5d. The next cut-off frequency is given by eqn $(30)_1^+$ and the corresponding eigenfunction (59) is shown in Fig. 5e.

5. LIMITING CASE

Thickness-twist motion

When the radius of the cylinder approaches infinity, the tangential displacement in the limit approaches a Cartesian form, and for large values of the argument, the Bessel functions approach trigonometric functions. The torsional motion of the laminated cylinder can then be identified with the thickness-twist motion of periodically laminated plates of thickness l and l' and material constants μ, ρ and μ', ρ', respectively. If we define new variables $\bar{\lambda}$, $\bar{\chi}$, $\bar{\Omega}$ and $\bar{\xi}$ for an isotropic plate of thickness l, according to the relations

$$\Omega = \frac{2a}{l}\tilde{\Omega}, \qquad \tilde{\Omega} = \omega/\omega_p, \qquad \omega_p = \frac{\pi}{l}(\mu/\rho)^{1/2},$$
$$\chi = \frac{2a}{l}\tilde{\chi}, \qquad \xi = \frac{2a}{l}\tilde{\xi}, \qquad \lambda = \frac{2a}{l}\tilde{\lambda}, \tag{64}$$

then

$$\tilde{\xi} = \sqrt{(\tilde{\Omega}^2 - \tilde{\chi}^2)}, \qquad \tilde{\xi} = \sqrt{[(\sigma\tilde{\Omega})^2 - \tilde{\chi}^2]}, \tag{65}$$

where $\tilde{\xi}$ and $\tilde{\chi}$ are the wave numbers normal and parallel to the laminations, respectively. The reference frequency ω_p is the lowest antisymmetric thickness-shear frequency of a plate of thickness l. The frequency eqn (25) now takes the form

$$4\mu\mu'\tilde{\xi}\tilde{\xi}' \cos \pi(1+q)\tilde{\lambda} + (\mu\tilde{\xi} - \mu'\tilde{\xi}')^2 \cos \pi(\tilde{\xi} - q\tilde{\xi}') - (\mu\tilde{\xi} + \mu'\tilde{\xi}')^2 \cos \pi(\tilde{\xi} + q\tilde{\xi}') = 0, \tag{66}$$

where $q = l'/l$ and $\tilde{\lambda}$ is now the Floquet exponent for the thickness-twist motion of the laminated plates. One can easily rewrite these equations in the form of eqns (26) which at each end of the zone factor into two equations.

For waves propagating normal to the layers $\chi = 0$, $\tilde{\xi} = \tilde{\Omega}$, $\tilde{\xi}' = \sigma\tilde{\Omega}$ and eqn (66) takes the simple form

$$4\sigma \cos \pi(1+q)\tilde{\lambda} + \left(\sqrt{\frac{\mu}{\mu'}} - \sigma\sqrt{\frac{\mu'}{\mu}}\right)^2 \cos \pi(1-\sigma q)\tilde{\Omega} - \left(\sqrt{\frac{\mu}{\mu'}} + \sigma\sqrt{\frac{\mu'}{\mu}}\right)^2 \cos \pi(1+\sigma q)\tilde{\Omega} = 0. \tag{67}$$

At the end-point $\tilde{\lambda} = 0$ of the first Brillouin zone, eqn (71) can be factored into the form

$$\mu\left(\tan\frac{\pi}{2}\Omega\right)^{\pm 1} + \sigma\mu'\left(\tan\frac{\pi}{2}\sigma q\Omega\right)^{\pm 1} = 0, \tag{68$_1$}$$

and at the other end-point of the zone $\tilde{\lambda} = 1/(1+q)$, and eqn (67) takes the form

$$\mu\left(\tan\frac{\pi}{2}\Omega\right)^{\pm 1} - \sigma\mu'\left(\cot\frac{\pi}{2}\sigma q\Omega\right)^{\pm 1} = 0. \tag{68$_2$}$$

Detailed analysis of the thickness-twist motion can be followed along the same lines as in the case of torsional motion of cylinders and is not further pursued here, but rather deferred to a subsequent study.

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